

Self-Gravitating Static Black Holes in $4D$ Einstein-Klein-Gordon System with Nonminimal Derivative Coupling

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Abstract

We study static black holes of four dimensional gravitational model with non-minimal derivative coupling and a scalar potential turned on. By taking an ansatz, namely, the first derivative of the scalar field is proportional to square root of a metric function, we reduce the Einstein field equation and the scalar field equation of motions into a single highly nonlinear differential equation. We show that near boundaries the solution is not the critical point of the scalar potential and the effective geometry becomes a space of constant scalar curvature.

1 Introduction

Modified Einstein's gravitational theories have been intensely studied over the last thirty years since they might uncover some cosmological problems such as the existence of dark matter and dark energy, the inflationary scenario in the early universe, and the accelerated expansion of our universe. Among the models, it is of interest to consider a model called nonminimal derivative coupling (NMDC) gravitational theory in four dimensions since it has been shown that this model provides an inflationary accelerated expansion model of our universe without introducing any scalar potential [1]. The action of the theory has the form [1, 2]

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left[\frac{R - 2\Lambda}{\kappa^2} - (\varepsilon g_{\mu\nu} + \xi R g_{\mu\nu} + \eta R_{\mu\nu}) \partial^\mu \phi \partial^\nu \phi \right] \quad (1.1)$$

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where $g_{\mu\nu}$ is a spacetime metric and $g \equiv \det(g_{\mu\nu})$, while $R_{\mu\nu}$ and R are the Ricci tensor and the scalar curvature, respectively. In the rest of this paper we use the indices $\alpha, \beta, \mu, \nu = 0, \dots, 3$. Also, we have a real scalar field ϕ . The constants ξ and η are real and called the derivative coupling parameter with dimension of the length-squared, whereas Λ is the cosmological constant and $\kappa \equiv 1/m_p$ where m_p is the Planck mass. The parameter $\varepsilon = \pm 1$, where in the case of $\varepsilon = 1$ we have a theory with a canonical scalar field, while $\varepsilon = -1$ we have a phantom scalar field [3].

The aim of this paper is to consider a class of static black hole solutions of four dimensional Einstein-scalar theory with nonminimal derivative coupling and the scalar potential $V(\phi)$ turned on. The spacetime metric is conformal to conformal to $\mathbf{M}^2 \times \mathbf{S}_k^2$ where \mathbf{M}^2 is a two-surface, while \mathbf{S}_k^2 are Einstein surfaces with $k = 0, \pm 1$. The latter cases are related to two-torus, two-sphere, and Riemann surfaces as discussed in the next section. To solve the Einstein field equation and the scalar equation of motions, we take an ansatz on the first derivative of the scalar field such that it is proportional to square root of a metric function, see (3.1). Such a setup simplifies the Einstein field equation and the scalar field equation of motions into a single highly nonlinear differential equation in Y where Y is a metric function. Although, it is difficult to solve this equation, it is of interest to focus the behavior of the solutions near the boundaries, namely, in the asymptotic region and near horizon limit.

In the asymptotic limit, we may have that at the zeroth order the function Y is fixed, but the scalar ϕ cannot be frozen. This implies that the scalar potential cannot be extremized. The "effective" geometry turns to be a space of constant scalar curvature but not maximally symmetric. This space could be anti-de Sitter (AdS) or a space with negative constant scalar curvature.

Near the horizon limit as the surfaces \mathbf{S}_k^2 become minimal or $Y = \epsilon$ with $0 < \epsilon \ll 1$. This would lead the blow up of the ansatz (3.1) unless the first derivative of a metric function $\rho(r)$ with respect to the radial coordinate r is proportional to $\epsilon^{\frac{1}{2}}$ where $\rho(r)$ can be thought of as the "radius" of \mathbf{S}_k^2 . The near-horizon geometry in this case is $\mathbb{R}^{1,1} \times \mathbf{S}_k^2$ where $\mathbb{R}^{1,1}$ is the 2-Minkowskian surface for $k = \pm 1$. The scalar field ϕ in this case depends linearly on r which implies that it is not the critical point of the scalar potential. This situation differs from the minimal derivative coupling model considered, for example, in [4, 5] and the nonminimal derivative coupling model discussed in [6].

The structure of the paper can be mentioned as follows. In section 2 we give a short review on gravitational theory with NMDC and the scalar potential. We also shortly discuss the Einstein field equation and the scalar equation of motions on static spacetimes. In section 3 we consider a special exact solution by setting the ansatz mentioned above and then, derive a highly nonlinear differential equation called master equation. Then, we discuss the behavior of the master equation near the boundaries in section 4. Finally, we conclude our results in Section 5.

2 A Short Review on NMDC Gravitational Theory

2.1 General Setup

In this subsection, we give a quick review on four dimensional NMDC Einstein-Klein-Gordon theory with scalar potential turned on. Such a theory equation is described by

the following action

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left[\frac{1}{\kappa^2} R - (\varepsilon g_{\mu\nu} + \eta G_{\mu\nu}) \partial^\mu \phi \partial^\nu \phi - 2V(\phi) \right], \quad (2.1)$$

where $G_{\mu\nu}$ is the Einstein tensor whose form is given by

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \quad (2.2)$$

and $V(\phi)$ is the real scalar potential. In this case, the pre-coefficients η and ξ have been chosen as $-2\xi = \eta$.

Varying (2.1) with respect to the metric $g_{\mu\nu}$, we get the Einstein field equation

$$G_{\mu\nu} = \kappa^2 (T_{\mu\nu}^{(\phi)} + T_{\mu\nu}^{(\eta)}), \quad (2.3)$$

with

$$\begin{aligned} T_{\mu\nu}^{(\phi)} &= \varepsilon \nabla_\mu \phi \nabla_\nu \phi - \frac{\varepsilon}{2} g_{\mu\nu} \nabla^\alpha \phi \nabla_\alpha \phi - g_{\mu\nu} V(\phi), \\ \frac{1}{\eta} T_{\mu\nu}^{(\eta)} &= R_{\mu\alpha} \nabla_\nu \phi \nabla^\alpha \phi + R_{\alpha\nu} \nabla^\alpha \phi \nabla_\mu \phi + \frac{1}{2} g_{\mu\nu} \nabla^\alpha \nabla^\beta \phi \nabla_\alpha \nabla_\beta \phi + \frac{1}{2} g_{\mu\nu} (\nabla_\alpha \nabla^\alpha \phi)^2 \\ &\quad - \frac{1}{2} G_{\mu\nu} \nabla^\alpha \phi \nabla_\alpha \phi - \nabla_\mu \nabla_\nu \phi \nabla_\alpha \nabla^\alpha \phi + R_{\mu\alpha\nu\beta} \nabla^\alpha \phi \nabla^\beta \phi + \nabla_\mu \nabla^\alpha \phi \nabla_\nu \nabla_\alpha \phi \\ &\quad - \frac{1}{2} R \nabla_\mu \phi \nabla_\nu \phi - g_{\mu\nu} \nabla^\alpha \nabla^\beta \phi \nabla_\alpha \nabla_\beta \phi - g_{\mu\nu} R^{\alpha\beta} \nabla_\alpha \phi \nabla_\beta \phi. \end{aligned} \quad (2.4)$$

The scalar field equation of motions can be obtained by varying (2.1) with respect to ϕ , namely

$$(\varepsilon g^{\mu\nu} + \eta G^{\mu\nu}) \nabla_\mu \nabla_\nu \phi = \frac{dV(\phi)}{d\phi}, \quad (2.5)$$

where we have used the fact $\nabla_\mu G^{\mu\nu} = 0$.

2.2 Static Spacetimes

In this subsection we particularly focus on a case of static spacetimes whose metric has the form

$$ds^2 = -f(r)dt^2 + g(r)dr^2 + \rho^2(r)d\Omega_k^2, \quad (2.6)$$

where r is the radial coordinate and $d\Omega_k^2$ describes two dimensional Einstein surfaces with $k = 0, \pm 1$. In the cases of $k = -1$, $k = 0$, and $k = 1$, we have Riemann surfaces, the two-torus, and the two-sphere, respectively. Note that among the functions $f(r)$, $g(r)$, and $\rho(r)$, only two of them are independent. This can be seen by employing a redefinition of the radial coordinate r to absorb one of them. It is important to write down the norm of the Riemann tensor related to the metric (2.6), namely

$$R^{\alpha\beta\mu\nu} R_{\alpha\beta\mu\nu} = \frac{3}{2f^2g^2} \left(f'' - \frac{f'^2}{2f} - \frac{f'g'}{2g} \right)^2 + \frac{3\rho'^2 f'^2}{\rho^2 f^2 g^2} + \frac{12}{\rho^2 g^2} \left(\rho'' - \frac{\rho'g'}{2g} \right)^2 + \frac{6}{\rho^4} \left(k - \frac{\rho'^2}{g} \right)^2, \quad (2.7)$$

which will be useful for our analysis in the next section. Moreover, throughout this paper we simply assume that the scalar field ϕ in (2.2) - (2.5) depends only on r , namely $\phi = \phi(r)$.

The above setup simplifies the field equations (2.2), namely

$$\begin{aligned}
\frac{\rho'g'}{\rho fg^2} - \frac{\rho'^2}{\rho^2 fg} - \frac{2\rho''}{\rho fg} + \frac{k}{\rho^2 f} &= \eta\kappa^2 \left(\frac{3\rho'g'}{2\rho fg^3} - \frac{\rho'^2}{2\rho^2 fg^2} - \frac{k}{2\rho^2 fg} - \frac{\rho''}{\rho fg^2} \right) \phi'^2 \\
&\quad + \frac{\varepsilon\kappa^2}{2} \frac{1}{fg} \phi'^2 - \eta\kappa^2 \frac{2\rho'}{\rho fg^2} \phi' \phi'' + \kappa^2 \frac{V}{f} , \\
\frac{\rho'f'}{\rho f} + \frac{\rho'^2}{\rho^2} - \frac{kg}{\rho^2} &= \frac{1}{2} \kappa^2 \varepsilon \phi'^2 - \kappa^2 gV + \eta\kappa^2 \left(\frac{3\rho'^2}{2\rho^2 g} + \frac{3\rho'f'}{2\rho fg} - \frac{k}{2\rho^2} \right) \phi'^2 \\
\frac{f''}{2f^2g} - \frac{f'g'}{4f^2g^2} - \frac{f'^2}{4f^3g} + \frac{\rho'f'}{2\rho f^2g} - \frac{\rho'g'}{2\rho fg^2} + \frac{\rho''}{\rho fg} &= -\frac{\varepsilon\kappa^2}{2} \frac{1}{fg} \phi'^2 - \kappa^2 \frac{V}{f} \\
&\quad + \eta\kappa^2 \left(\frac{f'}{2f^2g^2} + \frac{\rho'}{\rho fg^2} \right) \phi' \phi'' \\
&\quad + \eta\kappa^2 \left(\frac{f''}{4f^2g^2} - \frac{f'^2}{8f^3g^2} - \frac{3f'g'}{8f^2g^3} \right. \\
&\quad \left. + \frac{\rho''}{2\rho fg^2} + \frac{\rho'f'}{4\rho f^2g^2} - \frac{3\rho'g'}{4\rho fg^3} \right) \phi'^2
\end{aligned} \tag{2.8}$$

and the scalar field equation of motions (2.5), namely

$$\frac{1}{\rho^2 \sqrt{fg}} \left\{ \frac{\sqrt{fg}}{g} \left[\varepsilon \rho^2 + \eta \left(\frac{\rho \rho' f'}{fg} + \frac{\rho'^2}{g} - k \right) \right] \phi' \right\}' = \frac{dV(\phi)}{d\phi} , \tag{2.9}$$

where we have defined $A' \equiv dA/dr$ and $A'' \equiv d^2A/dr^2$ for any $A \equiv A(r)$.

3 A Special Class of Solutions

In this section we discuss a simple model where the scalar field has the form

$$\phi' = \nu g^{1/2} , \tag{3.1}$$

where ν is a non zero real constant and is determined by the field equations in (2.8) and (2.9). The ansatz (3.1) is inspired by the scalar-torsion theory [7] which can be viewed as the solutions of its field equations of motions. As we will see in the next section, this setup has a black hole solution which is different to the case of, for example, [4, 5].

Let us first discuss some consequences of (3.1). The condition (3.1) simply casts (2.8) and (2.9) into

$$\begin{aligned}
\frac{\rho'}{\rho} \left(\frac{g'}{g} - \frac{\rho'}{\rho} \right) - \frac{kg}{\rho^2} - \frac{2\rho''}{\rho} &= \frac{\kappa^2}{2 - \eta\kappa^2\nu^2} \left(\varepsilon\nu^2 + 2V - \frac{4k}{\kappa^2\rho^2} \right) g , \\
\frac{2 - 3\eta\nu^2\kappa^2}{\kappa^2} \frac{\rho'}{\rho} \left(\frac{f'}{f} + \frac{\rho'}{\rho} \right) &= - \left(-\varepsilon\nu^2 + 2V - \frac{(2 - \eta\nu^2\kappa^2)k}{\kappa^2\rho^2} \right) g , \\
\frac{\rho'}{2\rho} \left(\frac{g'}{g} - \frac{f'}{f} \right) + \frac{f'}{4f} \left(\frac{g'}{g} + \frac{f'}{f} \right) - \frac{f''}{2f} - \frac{\rho''}{\rho} &= \frac{\kappa^2}{2 - \eta\kappa^2\nu^2} (\varepsilon\nu^2 + 2V) , \\
\left\{ \nu\sqrt{f} \left[\varepsilon\rho^2 + \eta \left(\frac{\rho\rho'f'}{fg} + \frac{\rho'^2}{g} - k \right) \right] \right\}' &= \rho^2 \sqrt{fg} \frac{dV(\phi)}{d\phi} .
\end{aligned} \tag{3.2}$$

Then, by employing the transformation $f' = \nu\sqrt{g}\dot{f}$ and $f'' = \nu^2 \left(\ddot{f}g + \frac{\dot{f}\dot{g}}{2} \right)$ where $\dot{f} \equiv \frac{df}{d\phi}$, $\ddot{f} \equiv \frac{d^2f}{d\phi^2}$, we can rewrite (3.2) as

$$\begin{aligned} \frac{\ddot{\rho}}{\rho} + \frac{\dot{\rho}^2}{2\rho^2} + \frac{k}{2\nu^2\rho^2} &= -\frac{\kappa^2}{2\nu^2(2-\eta\kappa^2\nu^2)} \left(\varepsilon\nu^2 + 2V - \frac{4k}{\kappa^2\rho^2} \right) , \\ \frac{\dot{\rho}}{\rho} \left(\frac{\dot{f}}{f} + \frac{\dot{\rho}}{\rho} \right) &= -\frac{\kappa^2}{\nu^2(2-3\eta\nu^2\kappa^2)} \left(-\varepsilon\nu^2 + 2V - \frac{(2-\eta\nu^2\kappa^2)k}{\kappa^2\rho^2} \right) , \\ \frac{\ddot{\rho}}{\rho} + \frac{\dot{f}}{f} \left(\frac{\dot{\rho}}{2\rho} - \frac{\dot{f}}{4f} \right) + \frac{\ddot{f}}{2f} &= \frac{\kappa^2}{\nu^2(2-\eta\kappa^2\nu^2)} (\varepsilon\nu^2 + 2V) , \\ \left\{ \sqrt{f}\rho^2 \left[\varepsilon + \eta\nu^2 \frac{\dot{\rho}}{\rho} \left(\frac{\dot{f}}{f} + \frac{\dot{\rho}}{\rho} \right) - \eta \frac{k}{\rho^2} \right] \right\}^\cdot &= \frac{\rho^2\sqrt{f}}{\nu^2} \dot{V} . \end{aligned} \quad (3.3)$$

Now, in order to get the exact solutions we have two steps of introducing new variables as follows [7]. First, we define

$$\begin{aligned} x &= \ln \rho , \\ y &= \frac{\dot{\rho}}{\rho} , \\ z &= \frac{(\rho f)^\cdot}{\rho f} , \end{aligned} \quad (3.4)$$

which implies that the set of equation in (3.3) becomes

$$\begin{aligned} \dot{y} + \frac{3}{2}y^2 + \frac{k}{2\nu^2}e^{-2x} + \frac{\kappa^2}{2\nu^2(2-\eta\kappa^2\nu^2)} \left(\varepsilon\nu^2 + 2V - \frac{4k}{\kappa^2}e^{-2x} \right) &= 0 , \\ zy &= \frac{\kappa^2}{\nu^2(2-3\eta\nu^2\kappa^2)} \left(\varepsilon\nu^2 - 2V + (2-\eta\nu^2\kappa^2)\frac{k}{\kappa^2}e^{-2x} \right) \equiv F , \\ \dot{z} + \frac{1}{2}z^2 - \frac{k}{2\nu^2}e^{-2x} + \frac{\kappa^2}{2\nu^2(2-\eta\kappa^2\nu^2)} \left(3\varepsilon\nu^2 + 6V + \frac{4k}{\kappa^2}e^{-2x} \right) &= 0 , \\ (yz)^\cdot - \frac{1}{\eta\nu^4}\dot{V} + \frac{k}{2\nu^2}(y-z)e^{-2x} + \frac{1}{2}(3y+z) \left(\frac{\varepsilon}{\eta\nu^2} + yz \right) &= 0 . \end{aligned} \quad (3.5)$$

In this variable, it is easy to see that the second equation in (3.5) can be viewed as a constraint, while the last equation in (3.5) is superfluous [7].

Second, we introduce again a set of new variables

$$\begin{aligned} Y &\equiv y^2 , \\ Z &\equiv z^2 , \end{aligned} \quad (3.6)$$

such that we could have

$$\begin{aligned} \frac{1}{2} \frac{dY}{dx} &= \dot{y} , \\ \frac{1}{2} \sqrt{\frac{Y}{Z}} \frac{dZ}{dx} &= \dot{z} , \end{aligned} \quad (3.7)$$

where $\varsigma = \pm 1$. Then, (3.5) can be rewritten as

$$\begin{aligned} \frac{dY}{dx} + 3Y + \frac{k}{\nu^2} e^{-2x} + \frac{\kappa^2}{\nu^2 (2 - \eta \kappa^2 \nu^2)} \left(\varepsilon \nu^2 + 2V - \frac{4k}{\kappa^2} e^{-2x} \right) &= 0, \\ \frac{\kappa^4}{\nu^4 (2 - 3\eta \kappa^2 \nu^2)^2} \left(\varepsilon \nu^2 - 2V + \frac{k}{\kappa^2} (2 - \eta \kappa^2 \nu^2) e^{-2x} \right)^2 &= YZ, \\ \varsigma \sqrt{YZ} \frac{dZ}{dx} + Z^2 + \frac{\kappa^2}{\nu^2 (2 - \eta \kappa^2 \nu^2)} \left(3\varepsilon \nu^2 + 6V + \frac{k}{\kappa^2} (2 + \eta \kappa^2 \nu^2) e^{-2x} \right) Z &= 0. \end{aligned} \quad (3.8)$$

It is important to mention that there is a crucial miscalculation in transforming the equation of motions (3.3) into the new variables (Y, Z) in ref. [7]. The source of the error is the missing prefactor $\varsigma \sqrt{\frac{Y}{Z}}$ in the second equation of (3.7). Thus, they obtained the wrong master equation as we will see below.

After some computations using the first and the second equations in (3.8) we get the following results

$$\begin{aligned} Z &= \frac{\tilde{\chi}^2}{\chi^2 Y} \left(\frac{dY}{dx} + 3Y - 2k\eta\chi\nu^2 e^{-2x} + 2\varepsilon\chi\nu^2 \right)^2, \\ V &= -\frac{1}{2\chi} \left(\frac{dY}{dx} + 3Y \right) - \frac{\varepsilon\nu^2}{2} - \frac{k}{2\kappa^2} (2 + \eta\kappa^2\nu^2) e^{-2x}, \end{aligned} \quad (3.9)$$

where χ and $\tilde{\chi}$ are constant defined as

$$\begin{aligned} \chi &\equiv \frac{\kappa^2}{\nu^2 (2 - \eta\nu^2\kappa^2)}, \\ \tilde{\chi} &\equiv \frac{\kappa^2}{\nu^2 (2 - 3\eta\nu^2\kappa^2)}, \end{aligned} \quad (3.10)$$

which are non-singular, *i.e.* $\eta\kappa^2\nu^2 \neq 2$ and $\eta\kappa^2\nu^2 \neq 2/3$ for $\eta > 0$. Inserting Z and V in (3.9) into the third equation in (3.8) shows that we have a highly non-linear differential equation, namely

$$\begin{aligned} 2\varsigma \left(\frac{d^2Y}{dx^2} + 3\frac{dY}{dx} + 4k\eta\chi\nu^2 e^{-2x} \right) + \frac{1}{Y} \left(\frac{\chi}{\tilde{\chi}} - \varsigma \frac{dY}{dx} \right) \left(\frac{dY}{dx} + 3Y - 2k\eta\chi\nu^2 e^{-2x} + 2\varepsilon\chi\nu^2 \right)^2 \\ - \frac{\chi}{\tilde{\chi}} \left(3\frac{dY}{dx} + 9Y + \frac{4k\chi}{\kappa^2} (2 + \eta\kappa^2\nu^2) e^{-2x} \right) = 0, \end{aligned} \quad (3.11)$$

which is quite complicated and difficult to solve analytically. The equation (3.11) is the master equation which differs from that of the scalar-tensor theory [7]. We also have

$$\begin{aligned} \left(\frac{dx}{d\phi} \right)^2 &= Y(x), \\ \left[\frac{d \ln(\rho f)}{dx} \right]^2 &= \frac{Z}{Y}, \end{aligned} \quad (3.12)$$

such that the metric (2.6) can be written as [7]

$$ds^2 = -f dt^2 + \frac{d\rho^2}{\nu^2 \rho^2 Y} + \rho^2 d\Omega_k^2, \quad (3.13)$$

which implies that both Y and Z must be definitely positive. In this case at hand, the norm of the Riemann tensor (2.7) simplifies to

$$\begin{aligned} R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu} &= \frac{3\nu^2\rho^2Y}{2f^2} \left(f_{\rho\rho} - \frac{f_\rho^2}{2f} + \frac{f_\rho}{2} \left(\frac{2}{\rho} + \frac{1}{Y} \right) \right)^2 + \frac{3f_\rho^2}{f^2} \nu^4 \rho^2 Y^2 \\ &\quad + 3\nu^4 \rho^2 Y^2 \left(\frac{2}{\rho} + \frac{1}{Y} \right)^2 + \frac{6}{\rho^4} (k - \nu^2 \rho^2 Y)^2 , \end{aligned} \quad (3.14)$$

where $f_\rho \equiv \frac{df}{d\rho}$ and $f_{\rho\rho} \equiv \frac{d^2f}{d\rho^2}$. The Ricci scalar is simply given by

$$R = \left(-9 - \sqrt{\frac{Z}{Y}} - \frac{2\rho}{Y} \frac{dY}{d\rho} + \frac{2k}{\nu^2 \rho^2 Y} \right) \nu^2 Y . \quad (3.15)$$

4 Solutions Near Boundaries

In this section we would like to show that in our model related to the ansatz (3.1), it is possible to have a physical black hole solution. This can be seen in two regions, namely the asymptotic region and the near-horizon region. However, as mentioned above, we have generally different situations to the case considered, for example, in [4, 5]. This is so because the scalar field are not frozen near the boundaries which implies that the scalar potential cannot be extremized.

4.1 Around Asymptotic Region

In the asymptotic limit, namely at $r \rightarrow +\infty$, we may assume $x \rightarrow +\infty$ such that at the zeroth order, the solution of (3.11), namely Y , tends to a positive constant Y_0 which is

$$Y_0 = -\frac{2\varepsilon\nu^2\chi\tilde{\chi}}{3(\tilde{\chi} - \varsigma_1\chi)} , \quad (4.1)$$

where $\varsigma_1 = \pm 1$. By solving the second equation in (3.12), the lapse function f in (3.13) becomes

$$f(\rho) = A_0 \rho^{3\varsigma_1 - 1} , \quad (4.2)$$

with A_0 is a non-zero constant. In this limit, the effective geometry (3.13) becomes a static spacetime of constant Ricci curvature with

$$R = -(9 + 3\varsigma_1)\nu^2 Y_0 , \quad (4.3)$$

which can be split into

$$R = \begin{cases} -12\nu^2 Y_0 & \text{if } \varsigma_1 = 1 \text{ (AdS)} \\ -6\nu^2 Y_0 & \text{if } \varsigma_1 = -1 \text{ (non-Einstein)} \end{cases} . \quad (4.4)$$

The norm (3.14) tends to a quadratic form in ρ , namely

$$R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu} = 3\nu^4 Y_0^2 \rho^2 , \quad (4.5)$$

which shows that in this limit the spacetime is not maximally symmetric. The word "effective" means that the scalar field ϕ is not frozen in the region, namely, at the lowest order, the solution of the first equation in (3.12) is given by

$$\phi(\rho) = \phi_0 + Y_0^{-1/2} \ln \rho , \quad (4.6)$$

where ϕ_0 is a constant. This follows that the scalar potential $V(\phi)$ cannot be extremized but it has a constant value at the zeroth order, namely

$$V_0 = -\frac{3Y_0}{2\chi} - \frac{\varepsilon\nu^2}{2} . \quad (4.7)$$

Next, we consider the first order of solutions in the asymptotic limit using the perturbative ansatz

$$Y = Y_0 + Y_1 , \quad (4.8)$$

with $|Y_0| \ll |Y_1|$ such that the master equation (3.11) can be simplified into the following linear equation

$$\begin{aligned} & 2\tilde{\chi} \frac{d^2 Y_1}{dx^2} + 3 \left(4\tilde{\chi} - \chi - \frac{2\nu^2 \chi^3}{\chi - \tilde{\chi}} \right) \frac{dY_1}{dx} - 18(\chi - \tilde{\chi}) Y_1 \\ & + 4\tilde{\chi} \left[(2 + \eta\kappa^2\nu^2) \frac{\chi^2 k}{\tilde{\chi}\kappa^2} - \eta\nu^2 \chi k \right] e^{-2x} + 4\tilde{\chi} \chi^2 \eta^2 \nu^4 k^2 e^{-4x} = 0 , \end{aligned} \quad (4.9)$$

where we have simply set $\varsigma = \varsigma_1 = 1$. Moreover, we also take $e^{-mx} Y_1(x) \rightarrow 0$ and $e^{-mx} dY_1/dx \rightarrow 0$ for $m \geq 2$ around $x \rightarrow +\infty$. A special class of solutions of (4.9) has the form

$$\begin{aligned} Y_1 = & C_1 S[\lambda_1] e^{\lambda_1 x} + D_1 e^{\lambda_2 x} - \frac{2 \left((2 + \eta\kappa^2\nu^2) \frac{\chi^2 k}{\tilde{\chi}\kappa^2} - \eta\nu^2 \chi k \right)}{(\lambda_1 + 2)(\lambda_2 + 2)} e^{-2x} \\ & - \frac{2\chi^2 \eta^2 \nu^4 k^2}{(\lambda_1 + 4)(\lambda_2 + 4)} e^{-4x} , \end{aligned} \quad (4.10)$$

where $C_1, D_1 \in \mathbb{R}$ with

$$\lambda_{1,2} = \frac{3}{4} \left(\frac{\chi}{\tilde{\chi}} - 4 + \frac{2\varepsilon\nu^2 \chi^3}{\tilde{\chi}(\chi - \tilde{\chi})} \pm \left[\left(\frac{\chi}{\tilde{\chi}} - 4 + \frac{2\varepsilon\nu^2 \chi^3}{\tilde{\chi}(\chi - \tilde{\chi})} \right)^2 + 16 \left(\frac{\chi}{\tilde{\chi}} - 1 \right) \right]^{1/2} \right) , \quad (4.11)$$

and

$$S[\lambda_1] = \begin{cases} 0 & \text{if } \lambda_1 > 0 \\ 1 & \text{if } \lambda_1 < 0 \end{cases} , \quad (4.12)$$

The values of λ_1 or λ_2 should be negative, since $\lim_{x \rightarrow +\infty} Y_1(x) \rightarrow 0$, and $-2 < \lambda_{1,2} < 0$. The solution (4.10) belongs to the following two cases. First, by simply taking $0 < \chi/\tilde{\chi} < 1$ and $\eta > 0$, the analysis on (4.11) results $\varepsilon = 1$ and

$$2 < \eta\kappa^2\nu^2 < \frac{2}{3}(1 + 4\kappa^2) . \quad (4.13)$$

In this case, both λ_1 and λ_2 are negative and we exclude the large limit κ case. Second, we simply take $\chi/\tilde{\chi} > 1$ and $\eta < 0$. Analyzing (4.11), we find $\varepsilon = 1$ and

$$2 - \frac{32}{9}\kappa^2 < \eta\kappa^2\nu^2 < 0, \quad (4.14)$$

and only λ_2 is negative.

Solving the second equation in (3.12), we obtain the first order lapse function f

$$f(\rho) = A_0 \rho^{3\varsigma_{\mathbf{S}_1} - 1 + \frac{\varepsilon\tilde{\chi}}{\chi Y_0}(Y_1 + k\eta\chi\nu^2\rho^{-2})}, \quad (4.15)$$

while the first equation gives us

$$\begin{aligned} \phi(\rho) &= \phi_0 + \int \frac{dx}{\sqrt{Y_0 + Y_1}} \\ &\approx \phi_0 + Y_0^{-1/2} \ln \rho - \frac{1}{2} Y_0^{-3/2} \left(\frac{C_1}{\lambda_1} S[\lambda_1] \rho^{\lambda_1} + \frac{D_1}{\lambda_2} \rho^{\lambda_2} \right. \\ &\quad \left. + \frac{\left((2 + \eta\kappa^2\nu^2) \frac{\chi^2 k}{\tilde{\chi}\kappa^2} - \eta\nu^2\chi k \right)}{(\lambda_1 + 2)(\lambda_2 + 2)} \rho^{-2} + \frac{\chi^2\eta^2\nu^4 k^2}{2(\lambda_1 + 4)(\lambda_2 + 4)} \rho^{-4} \right). \end{aligned} \quad (4.16)$$

From the second equation in (3.9), we get the first order scalar potential

$$\begin{aligned} V &= -\frac{3Y_0}{2\chi} - \frac{\varepsilon\nu^2}{2} - \frac{1}{2\chi} (C_1(\lambda_1 + 3)S[\lambda_1] \rho^{\lambda_1} + D_1(\lambda_2 + 3) \rho^{\lambda_2}) \\ &\quad + \left(\frac{(2 + \eta\kappa^2\nu^2) \frac{\chi^2 k}{\tilde{\chi}\kappa^2} - \eta\nu^2\chi k}{\chi(\lambda_1 + 2)(\lambda_2 + 2)} - \frac{k}{2\kappa^2} (2 + \eta\kappa^2\nu^2) \right) \rho^{-2} + \frac{\chi^2\eta^2\nu^4 k^2}{\chi(\lambda_1 + 4)(\lambda_2 - 4)} \rho^{-4}. \end{aligned} \quad (4.17)$$

4.2 Near Horizon

Now, let us consider the near-horizon geometry. First, we write down the mean curvature H of $\mathbf{S}_k^2 \subseteq \Sigma^3$

$$H = \nu Y^{1/2}, \quad (4.18)$$

where Σ^3 is the hypersurface at $t = \text{constant}$. On the horizon, the surfaces \mathbf{S}_k^2 have to be minimal for all k , namely $H = 0$, which follows $Y = 0$. In order to evade the blow up of the metric (3.13) and the ansatz (3.1), around the region the function $\rho(r)$ should have the form

$$\rho(r) = \rho(r_h) + \varepsilon^{\frac{1}{2}}(r - r_h), \quad (4.19)$$

where r_h is the radius of the horizon, $\rho(r_h) > 0$, and $Y(r \rightarrow r_h) = \varepsilon$, $0 < \varepsilon \ll 1$. From (3.9), we find

$$\rho(r_h) = (\varepsilon k \eta)^{1/2}, \quad (4.20)$$

which follows that it forbids the 2-torus (or $k = 0$). The two possible cases are $k = \varepsilon$ and $\eta > 0$ or $k = -\varepsilon$ and $\eta < 0$. In this limit the lapse function $f(r)$ becomes a constant and

the near-horizon geometry has to be $\mathbb{R}^{1,1} \times \mathbf{S}_k^2$ where $\mathbb{R}^{1,1}$ is the 2-Minkowskian surface and $k = \pm 1$ [8] whose scalar curvature is given by

$$R = \frac{2\varepsilon}{\eta} . \quad (4.21)$$

The scalar field has the form

$$\phi(r) = \phi_h + \frac{1}{\rho(r_h)}(r - r_h) , \quad (4.22)$$

with ϕ_h is a constant. The scalar potential V in (3.9) becomes constant, namely

$$V_h = -\frac{\varepsilon\nu^2}{\eta\kappa^2} (1 + \eta\kappa^2\nu^2) . \quad (4.23)$$

Since the scalar potential V is not extremized, then we have

$$R \neq -4V_h . \quad (4.24)$$

We close this section by making some remarks as follows. First, if the order of ε in (4.19) is p with $p \neq \frac{1}{2}$, then this would imply $\nu = \varepsilon^{p-\frac{1}{2}}$. For $p < \frac{1}{2}$, the scalar field ϕ and V_h would diverge, while the scalar field ϕ would be frozen and $V_h \rightarrow 0$ for $p > \frac{1}{2}$. The latter case turns out to be forbidden since it leads to the inconsistency (4.21) and the condition $R = -4V_h$. Second, it is also possible that the metric (3.13) does not have any minimal surface, *i.e.* $H \neq 0$. If this is the case then the spacetime could be smooth everywhere or it might have the naked singularity at the origin.

5 Conclusions

So far we have considered a family of static black holes in four dimensional Einstein-Klein-Gordon theory with nonminimal derivative coupling and the scalar potential is turned on. The black holes also have unusual topology, namely, the two-surfaces are maximally symmetric with $k = 0, \pm 1$, see (2.6). In particular, we take the simple model satisfying the ansatz (3.1) which simplifies the Einstein field equation (2.8) and the scalar field equation of motions (2.9) which are coupled and highly nonlinear. This setup further implies that these coupled differential equations reduce into a single highly nonlinear differential equation, namely equation (3.11).

First, we showed that in the asymptotic limit, the metric function Y introduced in (3.12) tends to constant Y_0 but the scalar ϕ are not fixed. The latter case follows that the scalar potential V in (2.1) are not extremized. The effective geometry is the space of negative constant scalar curvature. Then, near the horizon, in order to have a regular metric the metric function $\rho(r)$ should take the form (4.19). The near horizon geometry is $\mathbb{R}^{1,1} \times \mathbf{S}_k^2$ where $\mathbb{R}^{1,1}$ is the 2-Minkowskian surface and $k = \pm 1$. The scalar field in this case has the linear form with respect to the radial coordinates r and the scalar potential V is not extremized.

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References

- [1] L. Amendola, “Cosmology with nonminimal derivative couplings,” *Phys. Lett. B* **301**, 175 (1993) [gr-qc/9302010].
- [2] G. W. Horndeski, “Second-order scalar-tensor field equations in a four-dimensional space,” *Int. J. Theor. Phys.* **10**, 363 (1974).
- [3] R. V. Korolev and S. V. Sushkov, “Exact wormhole solutions with nonminimal kinetic coupling,” *Phys. Rev. D* **90**, 124025 (2014) [arXiv:1408.1235 [gr-qc]].
- [4] G. Gibbons, R. Kallosh, and B. Kol, “Moduli, Scalar Charges, and the First Law of Black Hole Thermodynamics,” *Phys. Rev. Lett.* **77**, (1996) 4992, [arXiv:hep-th/9607108].
- [5] B. E. Gunara, F. P. Zen, A. Suroso, Arianto and F. T. Akbar, “Some Aspects of Spherical Symmetric Extremal Dyonic Black Holes in 4d N=1 Supergravity,” *Int. J. Mod. Phys. A* **28**, 1350084 (2013) [arXiv:1012.0971 [hep-th]].
- [6] M. Rinaldi, “Black holes with non-minimal derivative coupling,” *Phys. Rev. D* **86**, 084048 (2012) [arXiv:1208.0103 [gr-qc]].
- [7] G. Kofinas, E. Papantonopoulos and E. N. Saridakis, “Self-Gravitating Spherically Symmetric Solutions in Scalar-Torsion Theories,” *Phys. Rev. D* **91**, no. 10, 104034 (2015) [arXiv:1501.00365 [gr-qc]].
- [8] H. K. Kunduri, J. Lucietti, H. S. Reall, *Class. Quant. Grav.* **24** (2007) 4169 [arXiv:gr-qc/0705.4214].